

BERGMAN REPRESENTATIVE COORDINATES ON THE SIEGEL-JACOBI DISK

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ABSTRACT. We underline some differences between the geometric aspect of Berezin's approach to quantization on homogeneous Kähler manifolds and Bergman's construction for bounded domains in \mathbb{C}^n . We construct explicitly the Bergman representative coordinates for the Siegel-Jacobi disk \mathcal{D}_1^J , which is a partially bounded manifold whose points belong to $\mathbb{C} \times \mathcal{D}_1$, where \mathcal{D}_1 denotes the Siegel disk. The Bergman representative coordinates on \mathcal{D}_1^J are globally defined, the Siegel-Jacobi disk is a normal Kähler homogeneous Lu Qi-Keng manifold, whose representative manifold is the Siegel-Jacobi disk itself.

CONTENTS

1. Introduction	2
2. Berezin's quantization: the starting point	3
3. The geometry of the Siegel-Jacobi disk	9
3.1. The symmetric Fock space	10
3.2. Two-forms	11
3.3. Embeddings	14
4. Bergman representative coordinates on homogeneous Kähler manifolds	15
4.1. Definition	15
4.2. The simplest example: the Siegel disk	17
4.3. The Siegel-Jacobi disk	18
5. Appendix	21
5.1. Bergman pseudometric and Bergman metric	21
5.2. Embeddings	23
References	25

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1. INTRODUCTION

In this paper we discuss several geometric issues of Berezin's approach to quantization on homogeneous Kähler manifolds. The starting point is a weighted Hilbert space of square integrable holomorphic functions \mathfrak{H}_f defined on a homogeneous Kähler manifold M of weight e^{-f} , where the Kähler potential f is the logarithm of the kernel function K_M , which is obtained as the scalar product of two coherent states (CS) defined on M [57]. The metric ds_M^2 associated with $f = \ln K_M$ is in fact the so called *balanced metric* [34, 1, 51]. The metric ds_M^2 is different from the Bergman metric $ds_{\mathcal{B}_2}^2$, defined on bounded domains [20, 21, 22] or in Kobayashi's extension to manifolds, $ds_{\mathcal{B}_n}^2$, based on a Hilbert space of square integrable top degree holomorphic forms $\mathcal{F}_n(M)$ defined on M [43].

In a series of papers starting with [7, 8, 13] we have constructed CS [57] attached to the Jacobi group $G_n^J = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, where H_n denotes the $(2n+1)$ -dimensional Heisenberg group [35, 24]. The Jacobi group is important in Quantum Mechanics, being responsible for the squeezed states in Quantum Optics, see references in [7, 8, 9, 10, 11, 12]. The homogeneous Kähler manifolds \mathcal{D}_n^J are attached to the Jacobi group G_n^J [61, 11, 69]. The Siegel-Jacobi ball \mathcal{D}_n^J is not a bounded domain, but *partially bounded* (this denomination is borrowed from [68, 69]), its points being in $\mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n = \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}} / \mathrm{U}(n)$ denotes the Siegel bounded domain (Siegel ball) [8, 13, 68, 69].

We introduce the term *normal manifolds* of Lichnerowicz [49] to designate manifolds for which $K_M(z) \neq 0, \forall z \in M$. Also we use an extension of the meaning of *Kobayashi manifolds* which appears in the book [58] of Piatetski-Shapiro, and we advance the denomination *Lu Qi-Keng manifolds* to designate manifolds on which $K_M(z, \bar{w}) \neq 0, \forall z, w \in M$, extending to manifolds the notion introduced by Lu Qi-Keng for domains [52].

In this paper we construct explicitly the Bergman representative coordinates for the Siegel-Jacobi disk \mathcal{D}_1^J . Usually, the Bergman representative coordinates are defined on bounded domains $\mathcal{D} \subset \mathbb{C}^n$ [20]. We show that the Bergman representative coordinates on \mathcal{D}_1^J are globally defined and the Siegel-Jacobi disk is a homogeneous Kähler Lu Qi-Keng manifold, whose representative manifold is the Siegel-Jacobi disk itself.

The paper is laid out as follows. Berezin's recipe of quantization [16, 17, 18] using CS [57] adopted in [6, 7, 8, 13, 15] is presented in §2 in the geometric setting of the papers [34, 51, 1], where the ϵ -function [60, 27, 28] is constant. We underline the difference between the balanced metric and the Bergman metric [20, 21]. §3 summarizes the geometric information on the Siegel-Jacobi disk extracted from [7, 13, 14, 15] in a more precise formulation. In Proposition 1 we have included the expression of the Laplace-Beltrami operator on the Siegel-Jacobi disk, also obtained in [69]. In §3.3 we discuss the embedding of the Siegel-Jacobi disk in an infinite-dimensional projective Hilbert space, underling that the Jacobi group G_1^J is a CS-group [50, 56]. We emphasize the difference of this embedding comparatively with Kobayashi embedding [43]. §4 is devoted to the Bergman representative coordinates [20]. Firstly the definition of the Bergman representative coordinates for homogeneous Kähler manifolds is given. Then some general properties of the Bergman representative coordinates are recalled in Remark 3. In §4.2

the simplest example of the Siegel disk is presented. The notion of Lu Qi-Keng manifold is introduced. In [3] we have called the set $\Sigma_z := \{w \in M \mid K_M(z, \bar{w}) = 0\}$ *polar divisor of $z \in M$* , underling its meaning in algebraic geometry [40], and its equality with the cut locus [45] for some compact homogeneous manifolds. In Proposition 3 it is underlined that the Siegel-Jacobi disk is a Lu Qi-Keng manifold and the Bergman representative coordinates are globally defined on it. In Proposition 4 it is proved that the representative manifold of the Siegel-Jacobi disk is the Siegel-Jacobi disk itself. The Appendix §5.1 is dedicated to the Bergman pseudometric and metric. Also the notion of Kobayashi embedding [43] and Kobayashi manifold [58] are recalled in §5.2.

Notation. \mathbb{R} , \mathbb{C} , \mathbb{Z} and \mathbb{N} denotes the fields of real, complex numbers, the set of non-negative integers, respectively the ring of integers. We denote the imaginary unit $\sqrt{-1}$ by i , and the Real and Imaginary part of a complex number by \Re and respectively \Im , i.e. we have for $z \in \mathbb{C}$, $z = \Re z + i\Im z$, and $\bar{z} = \Re z - i\Im z$, also denoted $cc(z)$. In this paper the Hilbert space \mathfrak{H} is endowed with a scalar product (\cdot, \cdot) antilinear in the first argument, i.e. $(\lambda x, y) = \bar{\lambda}(x, y)$, $x, y \in \mathfrak{H}$, $\lambda \in \mathbb{C} \setminus 0$. We denote by d the differential, and we have $d = \partial + \bar{\partial}$, where, in a local system of coordinates (z_1, \dots, z_n) on a complex manifold M , we have $\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$.

2. BEREZIN'S QUANTIZATION: THE STARTING POINT

Firstly we highlight the relationship of the approach to geometry adopted in the papers [7, 8, 13, 14, 15] in the context of CS defined in mathematical physics [17, 57] with the “pure” geometric approach of mathematicians [20, 21, 22, 23, 43, 49, 1].

Let M be a complex manifold of complex dimension n . M is called *hermitian* if a hermitian structure H is given in its tangent bundle $T(M)$ [30]. If we choose a local coordinate system (z_1, \dots, z_n) , then a natural frame is given by $\frac{\partial}{\partial z_i}$, $i = 1, \dots, n$. Let us denote $h_{i\bar{j}} := H(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$, $i, j = 1, \dots, n$, and the matrix h is positive definite hermitian. A hermitian manifold is called *Kählerian* if its Kähler two-form, i.e. the real-valued $(1, 1)$ -form

$$(2.1) \quad \omega_M(z) = i \sum_{i,j=1}^n h_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j,$$

is closed, i.e. $d\omega_M = 0$. Equivalently (see e.g. Theorem C at p. 56 in [30]), *a hermitian manifold is Kählerian if and only if there exists a real-valued C^∞ function f - the Kähler potential - such that*

$$(2.2) \quad \omega_M = i\partial\bar{\partial}f,$$

and in (2.1) we have

$$(2.3) \quad h_{i\bar{j}}(z) = \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

An *automorphism* of the Kähler manifold M is an invertible holomorphic mapping preserving the Hermitian structure H . The Kähler manifold M is called *homogeneous* if the group $G(M)$ of all automorphisms of M acts transitively on it, see e.g. [38]. A “domain” is a connected open subset of \mathbb{C}^n , $\mathcal{D} \subset \mathbb{C}^n$, and “bounded” means relatively

compact. The Bergman metric (see [21, 37, 67] and (5.3) in the Appendix) defines in \mathcal{D} a canonical Kählerian structure, and $G_A(\mathcal{D}) = G(\mathcal{D})$, where $G_A(M)$ denotes the group of invertible holomorphic transformation of the manifold M [38].

Let us consider the weighted Hilbert space \mathfrak{H}_f of square integrable holomorphic functions on M , with weight e^{-f}

$$(2.4) \quad \mathfrak{H}_f = \left\{ \phi \in \text{hol}(M) \mid \int_M e^{-f} |\phi|^2 \Omega_M < \infty \right\},$$

where Ω_M is the volume form

$$(2.5) \quad \Omega_M := \frac{1}{n!} \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}},$$

G -invariant for G -homogeneous manifolds M .

If $\mathfrak{H}_f \neq 0$, let $\{\varphi_j(z)\}_{j=0,1,\dots}$ be an orthonormal base of functions of \mathfrak{H}_f and define the kernel function by

$$(2.6) \quad K_M(z, \bar{z}) = \sum_{i=0}^{\infty} \varphi_i(z) \bar{\varphi}_i(z).$$

For compact manifolds M , \mathfrak{H}_f is finite dimensional and the sum (2.6) is finite.

In order to fix the notation on CS [57], let us consider the triplet (G, π, \mathfrak{H}) , where π is a continuous, unitary, irreducible representation of the Lie group G on the separable complex Hilbert space \mathfrak{H} .

Let us now denote by H the isotropy group with Lie subalgebra \mathfrak{h} of the Lie algebra \mathfrak{g} of G . We consider (generalized) CS on complex homogeneous manifolds $M \cong G/H$ [57]. The *coherent vector mapping* φ is defined locally, on a coordinate neighborhood $\mathcal{V}_0 \subset M$ (cf. [5, 6]):

$$(2.7) \quad \varphi : M \rightarrow \bar{\mathfrak{H}}, \quad \varphi(z) = e_{\bar{z}},$$

where $\bar{\mathfrak{H}}$ denotes the Hilbert space conjugate to \mathfrak{H} . The vectors $e_{\bar{z}} \in \bar{\mathfrak{H}}$ indexed by the points $z \in M$ are called *Perelomov's CS vectors*. Using Perelomov's CS vectors, we consider Berezin's approach to quantization on Kähler manifolds with the supercomplete set of vectors verifying the Parceval overcompleteness identity [16]-[19],

$$(2.8) \quad (\psi_1, \psi_2)_{\mathfrak{H}^*} = \int_M (\psi_1, e_{\bar{z}})(e_{\bar{z}}, \psi_2) d\nu_M(z, \bar{z}), \quad \psi_1, \psi_2 \in \mathfrak{H},$$

where we have identified the space $\bar{\mathfrak{H}}$ complex conjugate to \mathfrak{H} with the dual space \mathfrak{H}^* of \mathfrak{H} . $d\nu_M$ in (2.8) is the quasi-invariant measure on M given by

$$(2.9) \quad d\nu_M(z, \bar{z}) = \frac{\Omega_M(z, \bar{z})}{(e_{\bar{z}}, e_{\bar{z}})},$$

where Ω_M is the normalized G -invariant volume form (2.5). In fact, (2.8), (2.9) and (2.5) are formula (2.1) and the next one on p. 1125 in Berezin's paper [17].

If the G -invariant Kähler two-form ω on the (real) $2n$ -dimensional manifold $M = G/H$ has the expression (2.1), then in (2.5) (see e.g. (4.2) in [2]) we have:

$$(2.10) \quad \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}} = i^n n! \mathcal{G}(z) \, d z_1 \wedge d \bar{z}_1 \wedge \dots \wedge d z_n \wedge d \bar{z}_n,$$

where the density of the G -invariant volume $\mathcal{G}(z)$ has the expression:

$$(2.11) \quad \mathcal{G}(z) := \det(h_{i\bar{j}}(z))_{i,j=1,\dots,n}.$$

If $z_j = x_j + iy_j$, $j = 1, \dots, n$, then we have the relations (see e.g. (5) on p. 14 in [67]):

$$(2.12) \quad 2^n x_1 \wedge y_1 \wedge \dots \wedge x_n \wedge y_n = i^n z_1 \wedge \bar{z}_1 \wedge \dots \wedge z_n \wedge \bar{z}_n = i^{n^2} z_1 \wedge \dots \wedge z_n \wedge \bar{z}_1 \wedge \dots \wedge \bar{z}_n.$$

We fix the orientation for which the real $2n$ -vector (2.12) is positive [67]. Introducing in (2.5) the relation (2.10), with (2.12), we find

$$(2.13) \quad \Omega_M = \mathcal{G} \, dV, \quad \text{where} \quad dV = 2^n d x_1 \wedge d y_1 \wedge \dots \wedge d x_n \wedge d y_n.$$

Let us introduce the mapping $\Phi : \mathfrak{H}^* \rightarrow \mathfrak{F}_{\mathfrak{H}}$ (cf [5, 6])

$$(2.14) \quad \Phi(\psi) := f_\psi, \quad f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\mathfrak{H}} = (e_{\bar{z}}, \psi)_{\mathfrak{H}}, \quad z \in \mathcal{V}_0 \subset M.$$

We denote by $\mathfrak{F}_{\mathfrak{H}} := L^2_{\text{hol}}(M, d\nu_M) \cap \mathcal{O}(M)$ the Hilbert space of holomorphic, square integrable functions with respect to the scalar product on M given by the r.h.s. of (2.8),

$$(2.15) \quad (f, g)_{\mathfrak{F}_{\mathfrak{H}}} = \int_M \bar{f}(z) g(z) \frac{\Omega_M(z, \bar{z})}{K_M(z, \bar{z})} = \int_M \bar{f}(z) g(z) \frac{\mathcal{G}}{K_M} \, dV,$$

where $K_M : M \times \bar{M} \rightarrow \mathbb{C}$ admits the local series expansion in a base of orthonormal functions $\{\varphi_i\}_{i \in \mathbb{N}}$ with respect with the scalar product (2.15), independent of the orthonormal base (cf Theorem 2.1 in [17], based on [36])

$$(2.16) \quad K_M(z, \bar{w}) \equiv (e_{\bar{z}}, e_{\bar{w}}) = \sum_{i=0}^{\infty} \varphi_i(z) \bar{\varphi}_i(w),$$

and

$$(2.17) \quad \int_M \bar{\varphi}_i \varphi_j \frac{\Omega_M}{K_M} = \delta_{ij}, \quad i, j \in \mathbb{N}.$$

The function $K_M(z, \bar{w})$ is a reproducing kernel, i.e. for $f \in \mathfrak{F}_{\mathfrak{H}}$, we have the relation

$$f(z) = (f, e_{\bar{z}})_{\mathfrak{F}_{\mathfrak{H}}} = \int_M (f, e_{\bar{w}})_{\mathfrak{F}_{\mathfrak{H}}} K_M(w, \bar{z}) \frac{\Omega_M(w, \bar{w})}{K_M(w, \bar{w})}.$$

The symmetric Fock space $\mathfrak{F}_{\mathfrak{H}}$ is the reproducing Hilbert space with the scalar product (2.15) attached to the kernel function K_M and the evaluation map Φ defined by (2.14) extends to an isometry [6]

$$(2.18) \quad (\psi_1, \psi_2)_{\mathfrak{H}^*} = (\Phi(\psi_1), \Phi(\psi_2))_{\mathfrak{F}_{\mathfrak{H}}} = (f_{\psi_1}, f_{\psi_2})_{\mathfrak{F}_{\mathfrak{H}}} = \int_M \bar{f}_{\psi_1}(z) f_{\psi_2}(z) \, d\nu_M(z).$$

In order to identify the Hilbert space \mathfrak{H}_f defined by (2.4) with the Hilbert space with scalar product (2.8), we have to consider the ϵ -function [60, 27, 28]

$$(2.19) \quad \epsilon(z) = e^{-f(z)} K_M(z, \bar{z}).$$

If the Kähler metric on the complex manifold M is obtained by the Kähler potential via (2.2) is such that $\epsilon(z)$ is a positive constant, then the metric is called *balanced*. This denomination was firstly used in [34] for compact manifolds, then it was used in [1] for noncompact manifolds and also in [51] in the context of Berezin quantization on homogeneous bounded domains. Note that *Condition A*) on p. 1132 in Berezin's paper [17] is exactly the condition $\epsilon = ct$ in (2.19), where we have included the Planck constant h in the Kähler potential f .

In [6, 8, 14] it was considered the case $\epsilon = 1$, i.e. we have considered in (2.2) $f = \ln K$.

Under the condition $\epsilon = 1$ in (2.19), the *hermitian balanced metric* of M in local coordinates is obtained from the scalar product (2.15) of two CS vectors $(e_{\bar{z}}, e_{\bar{z}}) = K_M(z, \bar{z})$ as

$$(2.20) \quad ds_M^2(z) = \sum_{i,j=1}^n h_{i\bar{j}} dz_i \otimes d\bar{z}_j = \sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(K_M(z, \bar{z})) dz_i \otimes d\bar{z}_j,$$

with the associated Kähler two-form

$$(2.21) \quad \omega_M(z) = i \sum_{i,j=1}^n h_{i\bar{j}} dz_i \wedge d\bar{z}_j = \sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(K_M(z, \bar{z})) dz_i \wedge d\bar{z}_j.$$

We recall that in the footnote * on p. 1128 in Berezin's paper [17] it is mentioned that instead of the balanced Kähler two-form ω_M given by formula (2.21), it should be alternatively used the (Bergman) Kähler two-form

$$(2.22) \quad \omega_M^1 = i\partial\bar{\partial} \ln(\mathcal{G}),$$

firstly used in the case of bounded domains and extended by Kobayashi [43] to a class of manifolds, see (5.18) in Proposition 5 in the Appendix. \mathcal{G} is defined in (2.10), (2.11). In such a case, in (2.8) we should use instead of $d\nu_M$ given by (2.9), the expression $d\nu_M = (\omega_M^1)^n$. Berezin has used in [17] the balanced Kähler two-form (2.21) and not the Bergman two-form (2.22) because he was able to prove the correspondence principle only with the balanced metric given by (2.20).

Strictly speaking, the construction of the Bergman kernel function was advanced for bounded domains $\mathcal{D} \subset \mathbb{C}^n$. Firstly the case $n = 2$ was considered in [21, 22, 23]. In general, for complex manifolds M , (2.20) defines a *pseudometric* (in the meaning of [42], see also the Appendix). Berezin himself has applied his construction to quantization of \mathbb{C}^n and to the noncompact hermitian symmetric spaces [16, 17, 18], realized as the classical domains I-IV in the formulation of Hua [41]. In the construction of Berezin, a family of Hilbert spaces $\mathfrak{H}_\lambda := \mathfrak{H}_{\lambda f}$ indexed by a positive number λ is considered, where the weight $e^{-\lambda f}$ ($\lambda = \frac{1}{h}$) is introduced in formula (2.4) instead of e^{-f} .

The relation (2.15) can be interpreted in the language of geometric quantization [47], see also [4, 15]. The local approach of Berezin to quantization via CS was globalized by Rawnsley, Cahen and Gutt in a series of papers starting with [60, 27, 28]. In their approach, the Kähler manifold M is not necessarily a homogeneous one, but they have proved that the homogeneous case corresponds to $\epsilon(z) = ct$ in (2.19).

Together with the Kähler manifold (M, ω) , it is also considered the triple $\sigma = (\mathcal{L}, h, \nabla)$, where \mathcal{L} is a holomorphic line bundle on M , h is the hermitian metric on

\mathcal{L} and ∇ is a connection compatible with metric and the Kähler structure [4]. With respect to a local holomorphic frame for the line bundle, the metric can be given as

$$h(s_1, s_2)(z) = \hat{h}(z) \bar{\hat{s}}_1(z) \hat{s}_2(z),$$

where \hat{s}_i is a local representing function for the section s_i , $i = 1, 2$, and $\hat{h}(z) = (e_{\bar{z}}, e_{\bar{z}})^{-1}$. (2.15) is the local representation of the scalar product on the line bundle σ . The connection ∇ has the expression $\nabla = \partial + \partial \ln \hat{h} + \bar{\partial}$. The curvature of \mathcal{L} is defined as $F(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$, and locally $F = \bar{\partial} \partial \ln \hat{h}$ [30]. The Kähler manifold (M, ω) is *quantizable* if there exists a triple σ such that $F(X, Y) = -i\omega(X, Y)$ and we have (2.21).

The extension of the construction of the Bergman metric on bounded domains in \mathbb{C}^n [21, 22, 23] to Kähler manifolds is a subtle one and was considered by Kobayashi [43, 46]. In order to do this construction, instead of the Bergman function, Kobayashi has considered the *Bergman kernel form*, see (5.6) in the §5.1 in Appendix. We also recall that in the §4 of Ch 4 of his monograph [58], entitled “*Kobayashi manifolds*”, Piatetski-Shapiro used this term in order to designate a certain class of complex manifolds studied by Kobayashi [43], which have the characteristics of bounded domains in \mathbb{C}^n .

The square of the length of a vector $X \in \mathbb{C}^n$, measured in this metric at the point $z \in M$, is [33]

$$\tau_M^2(z, X) := \sum_{i,j=1}^n h_{i\bar{j}}(z) X_i \bar{X}_j.$$

If the length l of a piecewise C^1 -curve $\gamma : [0, 1] \ni t \mapsto \gamma(t) \in M$ is defined as

$$l(\gamma) := \int_0^1 \tau_M(\gamma(t), \gamma'(t)) dt,$$

then the *Bergman distance* between two points $z_1, z_2 \in M$ is

$$d_B(z_1, z_2) = \inf \{l(\gamma) : \gamma \text{ is a piecewise curve s.t. } \gamma(0) = z_1, \gamma(1) = z_2\}.$$

We denote the *normalized Bergman kernel* (the *two-point function* of M [3, 4]) by

$$(2.23) \quad \kappa_M(z, \bar{z}') := \frac{K_M(z, \bar{z}')}{\sqrt{K_M(z) K_M(z')}} = (\tilde{e}_{\bar{z}}, \tilde{e}_{\bar{z}'}) = \frac{(e_{\bar{z}}, e_{\bar{z}'})}{\|e_{\bar{z}}\| \|e_{\bar{z}'}\|}.$$

Introducing in the above definition the series expansion (2.6), with the Cauchy - Schwartz inequality, we have that

$$(2.24) \quad |\kappa_M(z, \bar{z}')| \leq 1, \text{ and } \kappa_M(z, \bar{z}) = 1.$$

In this paper, by the *Berezin kernel* $b_M : M \times M \rightarrow [0, 1] \in \mathbb{R}$ we mean:

$$(2.25) \quad b_M(z, z') := |\kappa_M(z, \bar{z}')|^2.$$

Note that

$$(2.26) \quad D_M(z, z') := -\ln b_M(z, z') = -2 \ln |(\tilde{e}_{\bar{z}}, \tilde{e}_{\bar{z}'})|$$

is *Calabi's diastasis* [26] expressed via the CS vectors [28].

Let $\xi : \mathfrak{H} \setminus 0 \rightarrow \mathbb{P}(\mathfrak{H})$ be the canonical projection $\xi(z) = [z]$. The *Fubini-Study metric* in the nonhomogeneous coordinates $[z]$ is the hermitian metric on \mathbb{CP}^∞ (see [43] for details)

$$(2.27) \quad ds^2|_{FS}([z]) = \frac{(dz, dz)(z, z) - (dz, z)(z, dz)}{(z, z)^2}.$$

The elliptic *Cayley distance* [29] between two points in the projective Hilbert space $\mathbb{P}(\mathfrak{H})$ is defined as

$$(2.28) \quad d_C([z_1], [z_2]) = \arccos \frac{|(z_1, z_2)|}{\|z_1\| \|z_2\|}.$$

The Fubini-Study metric (2.27) and the Cayley distance (2.28) are independent of the homogeneous coordinates z representing $[z] = \xi(z)$.

Let M be a homogeneous Kähler manifold $M = G/H$ to which we associate the Hilbert space of functions $\mathfrak{F}_{\mathfrak{H}}$ with respect to the scalar product (2.15). We consider manifolds M which are *CS-orbits*, i.e. which admit a holomorphic embedding $\iota_M : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty)$ [50, 56, 6]. Note that this embedding differs from the standard Kobayashi embedding recalled in Theorem 6 in the Appendix.

For the following assertions, see [15]:

Remark 1. *Let us suppose that the Kähler manifold M admits a holomorphic embedding*

$$(2.29) \quad \iota_M : M \hookrightarrow \mathbb{CP}^\infty, \quad \iota_M(z) = [\varphi_0(z) : \varphi_1(z) : \dots].$$

The balanced Hermitian metric (2.20) on M is the pullback of the Fubini-Study metric (2.27) via the embedding (2.29), i.e.:

$$(2.30) \quad ds_M^2(z) = \iota_M^* ds_{FS}^2(z) = ds_{FS}^2(\iota_M(z)).$$

The angle defined by the normalized Bergman kernel (2.23) can be expressed via the embedding (2.29) as function of the Cayley distance (2.28)

$$(2.31) \quad \theta_M(z_1, z_2) = \arccos |\kappa_M(z_1, \bar{z}_2)| = \arccos |(\tilde{e}_{z_1}, \tilde{e}_{z_2})_M| = d_C(\iota_M(z_1), \iota_M(z_2)).$$

We have also the relation

$$(2.32) \quad d_B(z_1, z_2) \geq \theta_M(z_1, z_2).$$

The following (Cauchy) formula is true

$$(2.33) \quad (\tilde{e}_{z_1}, \tilde{e}_{z_2})_M = (\iota_M(z_1), \iota_M(z_2))_{\mathbb{CP}^\infty}.$$

The Berezin kernel (2.25) admits the geometric interpretation via the Cayley distance as

$$(2.34) \quad b_M(z_1, z_2) = \cos^2 d_C(\iota_M(z_1), \iota_M(z_2)) = \frac{1 + \cos(2d_C(\iota_M(z_1), \iota_M(z_2)))}{2}.$$

Note that traditionally “the Kobayashi embedding” is realized usually by the metric (5.11) obtained using the top-degree holomorphic forms on M , see the Appendix.

3. THE GEOMETRY OF THE SIEGEL-JACOBI DISK

The Siegel-Jacobi disk is the 4-dimensional homogenous space

$$(3.1) \quad \mathcal{D}_1^J := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C} \times \mathcal{D}_1,$$

associated to the 6-dimensional Jacobi group $G_1^J = H_1 \ltimes \mathrm{SU}(1,1)$, where H_1 is the 3-dimensional Heisenberg group, and the Siegel disk \mathcal{D}_1 is realized as

$$(3.2) \quad \mathcal{D}_1 = \{w \in \mathbb{C}^1 \mid |w| < 1\}.$$

It is easy to see [7] that:

Remark 2. *The Bergman kernel function $K_k : \mathcal{D}_1 \times \bar{\mathcal{D}}_1 \rightarrow \mathbb{C}$ is*

$$(3.3) \quad K_k(w, \bar{w}') := (1 - w\bar{w}')^{-2k} = \sum_n^{\infty} f_{nk}(w) \bar{f}_{nk}(w'),$$

where

$$(3.4) \quad f_{nk}(w) := \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}} w^n, \quad w \in \mathcal{D}_1,$$

and the Siegel disk \mathcal{D}_1 has the Kähler two-form ω_k given by

$$(3.5) \quad -i\omega_k(w) = \frac{2k}{(1 - w\bar{w})^2} dw \wedge d\bar{w},$$

$\mathrm{SU}(1,1)$ -invariant to the linear fractional transformation

$$(3.6) \quad w_1 = g \cdot w = \frac{aw + b}{\delta}, \quad \delta = \bar{b}w + \bar{a}, \quad \mathrm{SU}(1,1) \ni g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1.$$

The action (3.6) is transitive, and the Siegel disk \mathcal{D}_1 is a Kähler homogeneous manifold. \mathcal{D}_1 is a symmetric space.

Let us introduce the notation $\mathcal{D}_1^J \ni \varsigma := (z, w) \in \mathbb{C} \times \mathcal{D}_1$. Following the methods of [7], we get the reproducing kernel function $K : \mathcal{D}_1^J \times \mathcal{D}_1^J \rightarrow \mathbb{C}$, $K_{k\mu}(\varsigma, \varsigma') := (e_{\bar{z}, \bar{w}}, e_{\bar{z}', \bar{w}'}),$ obtained from the scalar product of two CS vectors of the type $e_{z,w}$, where $k = k' + \frac{1}{4}$, $2k' \in \mathbb{Z}$ and $\mu \in \mathbb{R}_+$:

$$(3.7) \quad K_{k\mu}(\varsigma, \varsigma') = (1 - w\bar{w}')^{-2k} \exp \mu F(\varsigma, \varsigma'), \quad F(\varsigma, \varsigma') = \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2w}{2(1 - w\bar{w}')}.$$

In particular, the kernel on the diagonal, $K_{k\mu}(\varsigma) = (e_{\bar{z}, \bar{w}}, e_{\bar{z}, \bar{w}})$, reads

$$(3.8) \quad K_{k\mu}(\varsigma) = (1 - w\bar{w})^{-2k} \exp \mu F(z, w), \quad F(z, w) = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{2(1 - w\bar{w})},$$

and evidently $K_{k\mu}(\varsigma) > 0$, $\forall \varsigma \in \mathcal{D}_1^J$. The holomorphic, transitive and effective action of the Jacobi group G_1^J on the manifold \mathcal{D}_1^J is $(g, \alpha) \circ (z, w) \rightarrow (z_1, w_1)$, where w_1 is given by (3.6) and

$$(3.9) \quad z_1 = \frac{\gamma}{\delta}, \quad \gamma = z + \alpha - \bar{\alpha}w.$$

3.1. The symmetric Fock space. The scalar product (2.8) of functions from the space $\mathfrak{F}_{k\mu} = L^2_{\text{hol}}(\mathcal{D}_1^J, \rho_{k\mu})$ corresponding to the kernel $K_{k\mu}$ defined by (3.8) on the manifold (3.1) is [7, 15]:

$$(3.10) \quad (\phi, \psi)_{k\mu} = \int_{\mathcal{D}_1^J} \bar{f}_\phi(z, w) f_\psi(z, w) \rho_{k\mu} d\nu(z, w), \quad \rho_{k\mu} = \Lambda(1 - w\bar{w})^{2k} e^{-\mu \frac{2|z|^2 + z^2 \bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})}},$$

$$(3.11) \quad \Lambda = \frac{4k - 3}{2\pi^2}.$$

The value of the G_1^J -invariant measure (2.9), obtained in (3.25), is

$$(3.12) \quad d\nu(z, w) = \mu \frac{d\Re w d\Im w}{(1 - w\bar{w})^3} d\Re z d\Im z.$$

The base of orthonormal functions attached to kernel (3.7) defined on the manifold \mathcal{D}_1^J consists of the holomorphic polynomials [7], [9]

$$(3.13) \quad f_{nk'm}(z, w) = f_{mk'}(w) \frac{P_n(\sqrt{\mu}z, w)}{\sqrt{n!}}, \quad k = k' + \frac{1}{4}, \quad 2k' \in \mathbb{Z}_+, \quad \mu \in \mathbb{Z}_+,$$

where the monomials $f_{mk'}$ are defined in (3.4), while

$$(3.14) \quad P_n(z, w) = n! \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{w}{2}\right)^p \frac{z^{n-2p}}{p!(n-2p)!}.$$

The series expansion (2.6) of the kernel function (3.7) reads

$$(3.15) \quad K_{k\mu}(z, w; \bar{z}', \bar{w}') = \sum_{n,m=0}^{\infty} f_{nk'm}(z, w) \bar{f}_{nk'm}(z', w').$$

We may also write down the expression (3.15) as

$$(3.16) \quad K_{k\mu}(z, w; \bar{z}', \bar{w}') = \sum_{n,m=0}^{\infty} \tilde{f}_{nkm}(z, w) \bar{\tilde{f}}_{nkm}(z', w'),$$

where

$$(3.17) \quad \tilde{f}_{nkm}(z, w) = a_{nkm} w^m P_n(\sqrt{\mu}z, w), \quad a_{nkm} = \sqrt{\frac{\Gamma(m + 2k - 1)}{m! \Gamma(2k - 1) n!}},$$

and the orthonormality of the base can be written as (see calculation in [10])

$$(3.18) \quad (\tilde{f}_{nks}, \tilde{f}_{mkr})_{k\mu} = \delta_{nm} \delta_{rs}, \quad n, m, r, s \in \mathbb{N}, \quad k > \frac{1}{2}.$$

3.2. Two-forms. We recall the definitions of the notions which appear in Proposition 1.

The *Ricci form associated to the Kählerian two-form* ω_M (2.1) is (see p. 90 in [55])

$$(3.19) \quad \rho_M(z) := i \sum_{\alpha, \beta=1}^n \text{Ric}_{\alpha\bar{\beta}}(z) \, dz_\alpha \wedge d\bar{z}_\beta, \quad \text{Ric}_{\alpha\bar{\beta}}(z) = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln \mathcal{G}(z).$$

The scalar curvature at a point $p \in M$ of coordinates z is (see p. 294 in [44])

$$(3.20) \quad s_M(p) := \sum_{\alpha, \beta=1}^n (h_{\alpha\bar{\beta}})^{-1} \text{Ric}_{\alpha\bar{\beta}}(z).$$

We use the following expression for the Laplace-Beltrami operator on Kähler manifolds M with the Kähler two-form (2.1), cf e.g. Lemma 3 in the Appendix of [17]:

$$(3.21) \quad \Delta_M(z) := \sum_{\alpha, \beta=1}^n (h_{\alpha\bar{\beta}})^{-1} \frac{\partial^2}{\partial \bar{z}_\alpha \partial z_\beta}.$$

Following [43, 53], let us introduce also the positive definite (1,1)-form on $M = G/H$

$$(3.22) \quad \tilde{\omega}_M(z) := i \sum_{\alpha, \beta \in \Delta_+} \tilde{h}_{\alpha\bar{\beta}}(z) \, dz_\alpha \wedge d\bar{z}_\beta, \quad \tilde{h}_{\alpha\bar{\beta}}(z) := (n+1)h_{\alpha\bar{\beta}}(z) - \text{Ric}_{\alpha\bar{\beta}}(z),$$

which is Kähler, with Kähler potential

$$\tilde{f} := \ln(K(z)^{n+1} \mathcal{G}(z)).$$

Proposition 1. *The balanced Kähler two-form $\omega_{k\mu}$ on \mathcal{D}_1^J , G_1^J -invariant to the action (3.6), (3.9), can be written as:*

$$(3.23) \quad -i\omega_{k\mu}(z, w) = 2k \frac{dw \wedge d\bar{w}}{P^2} + \mu \frac{\mathcal{A} \wedge \bar{\mathcal{A}}}{P}, \quad \mathcal{A} = dz + \bar{\eta}(z, w) dw, \quad \eta(z, w) = \frac{z + \bar{z}w}{P},$$

where

$$P = 1 - w\bar{w}.$$

The balanced hermitian metric on \mathcal{D}_1^J corresponding to the Kähler two-form (3.23) is:

$$(3.24) \quad ds_{k\mu}^2(z, w) = 2k \frac{dw \otimes d\bar{w}}{P^2} + \mu \frac{\mathcal{A} \otimes \bar{\mathcal{A}}}{P}.$$

The volume form is:

$$(3.25) \quad \omega_{k\mu} \wedge \omega_{k\mu} = 16k\mu(P)^{-3} d\Re z \wedge d\Im z \wedge d\Re w \wedge d\Im w,$$

giving for the density of the G_1^J -invariant volume \mathcal{G} (2.11) of Siegel-Jacobi disk a value independent of z

$$(3.26) \quad \mathcal{G}_{\mathcal{D}_1^J}(z, w) = \frac{2k\mu}{(1 - w\bar{w})^3}.$$

The Bergman metric (5.14) is

$$(3.27) \quad ds_{\mathcal{B}}^2(z, w)_{\mathcal{D}_1^J} = 3 \frac{dw \otimes d\bar{w}}{(1 - w\bar{w})^2},$$

with the associated Bergman Kähler two-form (5.18)

$$(3.28) \quad \omega_{\mathcal{D}_1^J}^1(z, w) = 3i \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2}.$$

The Kähler two-form (3.22) for \mathcal{D}_1^J , corresponding to the Kähler potential

$$\tilde{f}(z, w) = 3[\mu f(z, w) - (2k + 1) \ln(1 - w\bar{w})],$$

reads

$$(3.29) \quad -i\tilde{\omega}_{\mathcal{D}_1^J}(z, w) = 3 \left[(2k + 1) \frac{dw \wedge d\bar{w}}{P^2} + \mu \frac{\mathcal{A} \wedge \bar{\mathcal{A}}}{P} \right].$$

The Ricci form (3.19) associated with the balanced metric (3.24) reads

$$(3.30) \quad \rho_{\mathcal{D}_1^J}(z, w) = -3i \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2},$$

and \mathcal{D}_1^J is not an Einstein manifold with respect to the balanced metric (3.24), but it is one with respect to the Bergman metric (3.27).

The scalar curvature (3.20) is constant and negative definite

$$(3.31) \quad s_{\mathcal{D}_1^J}(z, w) = -\frac{3}{2k}, \quad p \in \mathcal{D}_1^J.$$

The Laplace-Beltrami operator (3.21) on the Siegel-Jacobi disk has the expression

$$(3.32) \quad \begin{aligned} \Delta_{\mathcal{D}_1^J}(z, w) &= \frac{P^2}{2k\mu} \left[\left(\frac{2k}{P} + \mu|\eta|^2 \right) \frac{\partial^2}{\partial z \partial \bar{z}} + \mu \left(\frac{\partial^2}{\partial w \partial \bar{w}} - \bar{\eta} \frac{\partial^2}{\partial z \partial \bar{w}} - \eta \frac{\partial^2}{\partial \bar{z} \partial w} \right) \right] \\ &= \left(\frac{1 - w\bar{w}}{\mu} + \frac{|z + \bar{z}w|^2}{2k} \right) \frac{\partial^2}{\partial z \partial \bar{z}} + \\ &\quad + \frac{1 - w\bar{w}}{2k} \left[(1 - w\bar{w}) \frac{\partial^2}{\partial w \partial \bar{w}} - (\bar{z} + zw) \frac{\partial^2}{\partial z \partial \bar{w}} - (z + \bar{z}w) \frac{\partial^2}{\partial \bar{z} \partial w} \right], \end{aligned}$$

G_1^J -invariant to the action (3.6), (3.9), i.e.

$$(3.33) \quad \Delta_{\mathcal{D}_1^J}(z_1, w_1) = \Delta_{\mathcal{D}_1^J}(z, w).$$

Also we have the relations:

$$(3.34) \quad \Delta_{\mathcal{D}_1^J}(z, w) (\ln(\mathcal{G}_{\mathcal{D}_1^J}(z, w))) = -s_{\mathcal{D}_1^J}(z, w) = \frac{3}{2k}.$$

Proof. Most of the assertions of Proposition 1 have been proved in [7, 15]. Here we emphasize some differences between the balanced and Bergman metric on the Siegel-Jacobi disk. However, for self-containment, we indicate the main ingredients of the proof.

We calculate the Kähler potential on \mathcal{D}_1^J as the logarithm of the reproducing kernel $f(z, w) = \ln K_{k\mu}(z, w)$, $z, w \in \mathbb{C}$, $|w| < 1$, i.e.

$$(3.35) \quad f(z, w) = \mu \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{2(1 - w\bar{w})} - 2k \ln(1 - w\bar{w}).$$

The balanced Kähler two-form ω is obtained with formulas (2.1), (2.3), i.e.

$$(3.36) \quad -i\omega_{k\mu}(z, w) = h_{z\bar{z}}dz \wedge d\bar{z} + h_{z\bar{w}}dz \wedge d\bar{w} - h_{\bar{z}w}d\bar{z} \wedge dw + h_{w\bar{w}}dw \wedge d\bar{w}.$$

The volume form (2.5) for \mathcal{D}_1^J is:

$$(3.37) \quad -\omega \wedge \omega = 2 \begin{vmatrix} h_{z\bar{z}} & h_{z\bar{w}} \\ h_{\bar{z}w} & h_{w\bar{w}} \end{vmatrix} dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}.$$

The matrix of the balanced metric (2.20) $h = h(\varsigma)$, $\varsigma := (z, w) \in \mathbb{C} \times \mathcal{D}_1$, determined with the Kähler potential (3.35), reads

$$(3.38) \quad h(\varsigma) := \begin{pmatrix} h_{z\bar{z}} & h_{z\bar{w}} \\ h_{\bar{z}w} & h_{w\bar{w}} \end{pmatrix} = \begin{pmatrix} \frac{\mu}{P} & \mu \frac{\eta}{P} \\ \mu \frac{\bar{\eta}}{P} & \frac{2k}{P^2} + \mu \frac{|\eta|^2}{P} \end{pmatrix},$$

where η is defined as in [7]

$$(3.39) \quad z = \eta - w\bar{\eta}, \quad \text{and} \quad \eta = \eta(z, w) := \frac{z + \bar{z}w}{1 - w\bar{w}}.$$

The inverse of the matrix (3.38) reads

$$(3.40) \quad h^{-1}(\varsigma) = \frac{P^3}{2k\mu} \begin{pmatrix} \frac{2k}{P^2} + \mu \frac{|\eta|^2}{P} & -\mu \frac{\eta}{P} \\ -\mu \frac{\bar{\eta}}{P} & \frac{\mu}{P} \end{pmatrix}.$$

The check out the G_1^J -invariance of the Laplace-Beltrami operator (3.32) is an easy calculation, but quite long. We indicate some intermediate stages.

The inverse of the relations (3.6), (3.9) are

$$(3.41) \quad w = \frac{\bar{a}w_1 - b}{-\bar{b}w_1 + a}, \quad z = \frac{-z_1 + \alpha a + \bar{\alpha}b - w_1(\alpha\bar{b} + \bar{\alpha}a)}{-\bar{b}w_1 + a},$$

where $g \in \text{SU}(1, 1)$, defined by (3.6), and $\alpha \in \mathbb{C}$ define the action of the Jacobi group on the Siegel-Jacobi disk. We have

$$\frac{\partial}{\partial z_1} = \frac{\partial z}{\partial z_1} \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial w_1} = \frac{\partial w}{\partial w_1} \frac{\partial}{\partial w} + \frac{\partial z}{\partial w_1} \frac{\partial}{\partial z},$$

where, with (3.6), (3.9), (3.41), we find:

$$(3.42) \quad \begin{aligned} \frac{\partial z}{\partial z_1} &= \delta; \quad \frac{\partial w}{\partial w_1} = \delta^2; \quad \frac{\partial z}{\partial w_1} = \delta\vartheta, \quad \vartheta = \bar{\alpha}\bar{a} + \bar{b}(z + \alpha). \\ \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} &= |\delta|^2 \frac{\partial^2}{\partial z \partial \bar{z}}, \\ \frac{\partial^2}{\partial z_1 \partial \bar{w}_1} &= |\delta|^2 (\bar{\delta} \frac{\partial^2}{\partial z \partial \bar{w}} + \bar{\vartheta} \frac{\partial^2}{\partial z \partial \bar{z}}), \\ \frac{\partial^2}{\partial w_1 \partial \bar{w}_1} &= |\delta|^2 (|\delta|^2 s \frac{\partial^2}{\partial w \partial \bar{w}} + \bar{\delta}\vartheta \frac{\partial^2}{\partial z \partial \bar{w}} + \delta\bar{\vartheta} \frac{\partial^2}{\partial \bar{z} \partial w} + |\vartheta|^2 \frac{\partial^2}{\partial z \partial \bar{z}}). \end{aligned}$$

We also find easily

$$(3.43) \quad \begin{aligned} P' &= 1 - w_1\bar{w}_1 = \frac{P}{|\delta|^2}, \\ \eta_1 &= \frac{z_1 + \bar{z}_1 w_1}{P'} = a(\eta + \alpha) + b(\bar{\eta} + \bar{\alpha}). \end{aligned}$$

With (3.42), (3.43), we check out the invariance (3.33) of the Laplace-Beltrami operator (3.32) on the Siegel-Jacobi disk to the action of the Jacobi group G_1^J . \square

In Remark 2 in [13] we have already stressed that the Kähler two-form (3.23), firstly calculated in [7], is identical with the one obtained by J.-H. Yang, see e.g. Theorem 1.3 in [69], where A, B, w, η corresponds in our notation with respectively $\frac{1}{2}k, \frac{1}{4}\mu, -w, z$. Under the same correspondence, the formula (3.32) is a particular case of Theorem 1.4 in [69]. The scalar curvature (3.31) was previously obtained in [70]. We recall that Theorem 2.5 in Berezin's paper [17] asserts essentially that $\Delta_M(z)(\ln(\mathcal{G}(z))) = ct$ for the balanced metric plus other 3 conditions.

3.3. Embeddings. We recall that the homogeneous Kähler manifolds $M = G/H$ which admit an embedding in a projective Hilbert space as in Remark 1 are called *CS-manifolds*, and the corresponding groups G are called *CS-groups* [50, 56]. We particularize Remark 1 in the case of the Siegel-Jacobi disk and we have:

Proposition 2. *The Jacobi group G_1^J is a CS-group and the Siegel-Jacobi disk \mathcal{D}_1^J is a quantizable Kähler CS-manifold. The Hilbert space of functions $\mathfrak{F}_{\mathfrak{H}}$ is the space $\mathfrak{F}_{k\mu} = L_{\text{hol}}^2(\mathcal{D}_1^J, \rho_{k\mu})$ with the scalar product (3.10)-(3.12). The Kählerian embedding $\iota_{\mathcal{D}_1^J} : \mathcal{D}_1^J \hookrightarrow \mathbb{CP}^\infty$ (2.29) $\iota_{\mathcal{D}_1^J} = [\Phi] = [\varphi_0 : \varphi_1 : \dots \varphi_N : \dots]$ is realized with an ordered version of the base functions $\Phi = \{f_{nk'm}(z, w)\}$ given by (3.13), and the Kähler two-form (3.23) is the pullback of the Fubini-Study Kähler two-form (2.27) on \mathbb{CP}^∞ ,*

$$\omega_{k\mu} = \iota_{\mathcal{D}_1^J}^* \omega_{FS}|_{\mathbb{CP}^\infty}, \quad \omega_{k\mu}(z, w) = \omega_{FS}([\varphi_N(z, w)]).$$

The normalized Bergman kernel (2.23) $\kappa_{k\mu}$ of the Siegel-Jacobi disk expressed in the variables $\varsigma = (z, w)$, $\varsigma' = (z', w')$ reads

$$(3.44) \quad \kappa_{k\mu}(\varsigma, \varsigma') = \kappa_k(w, \bar{w}') \exp[\mu(F(\varsigma, \bar{\varsigma}') - \frac{1}{2}(F(\varsigma) + F(\varsigma')))],$$

where $\kappa_k(w, \bar{w}')$ is the normalized Bergman kernel for the Siegel disk \mathcal{D}_1

$$(3.45) \quad \kappa_k(w, \bar{w}') = \left[\frac{(1 - |w|^2)(1 - |w'|^2)}{(1 - w\bar{w}')^2} \right]^k,$$

$F(\varsigma, \bar{\varsigma}')$ is defined in (3.7), and $F(\varsigma)$ is defined in (3.8). The Berezin kernel of \mathcal{D}_1^J is

$$b_{k\mu}(\varsigma, \varsigma') = b_k(w, w') \exp[2\Re F(\varsigma, \varsigma') - F(\varsigma) - F(\varsigma')],$$

where $b_k(w, w') = |\kappa_k(w, \bar{w}')|^2$.

With formula (2.26), we get for the diastasis function on the Siegel-Jacobi disk the expression:

$$(3.46) \quad \frac{D_{k\mu}(\varsigma, \varsigma')}{2} = k \ln \frac{|1 - w\bar{w}'|^2}{(1 - |w|^2)(1 - |w'|^2)} + \mu \left[\frac{F(\varsigma) + F(\varsigma')}{2} - \Re F(\varsigma, \bar{\varsigma}') \right].$$

Proof. The above proposition was enunciated in [15]. Here we use the notion of embedding as in the standard definition recalled at the beginning of §5.2. This approach is different from the standard Kobayashi embedding summarized in §5.2.2 of the Appendix. We use the explicit representation (3.16). In accord with (3.17), we have

$$\tilde{f}_{0k0} = 1, \quad \tilde{f}_{0k1} = \sqrt{2k-1}w, \quad \tilde{f}_{1k0} = \sqrt{\mu}z,$$

$$\frac{\partial(\tilde{f}_{0k1}, \tilde{f}_{1k0})}{\partial(z, w)} = -\sqrt{\mu(2k-1)} \neq 0, \quad k > \frac{1}{2}, \quad \mu > 0,$$

and we get

$$K_{k\mu}(z, \bar{w}) = 1 + (2k-1)|w|^2 + \mu|z|^2 + \sum_{n,s \in G} |\tilde{f}_{nks}(z, w)|^2,$$

where $G = \mathbb{N} \times \mathbb{N} \setminus \{(0, 1) \cup (1, 0)\}$. □

4. BERGMAN REPRESENTATIVE COORDINATES ON HOMOGENEOUS KÄHLER MANIFOLDS

4.1. Definition. Bergman has introduced the representative coordinates on bounded domains $\mathcal{D} \subset \mathbb{C}^n$ [20] in order to generalize the Riemann mapping theorem to \mathbb{C}^n , $n > 1$. However, almost the same construction works in the case of manifolds M [32, 33] instead of bounded domains. Berezin's approach to quantization [16, 17, 18, 19] recalled in §2 was applied to manifolds M which are (symmetric) bounded domains $\mathcal{D} \subset \mathbb{C}^n$ (in fact, hermitian symmetric spaces) and \mathbb{C}^n . If the same construction is applied to manifolds M which are not necessarily bounded domains, then the Hilbert space $\mathfrak{F}_{\mathfrak{H}}$ (2.15) usually is replaced with the Hilbert space of square integrable global holomorphic forms of top degree $\mathcal{F}_n(M)$ defined by (5.5) in the Appendix. In the presentation below of the Bergman representative coordinates we shall use the notation for bounded domains [33] adapted for manifolds M .

Assume now that (2.21) defines a Kählerian structure on the complex n -dimensional manifold M . Then

$$(4.1) \quad \mathcal{G}(z) = \det h_{i,\bar{j}}(z) = \det \frac{\partial^2 \ln(K_M(z, \bar{z}))}{\partial z_i \partial \bar{z}_j} \neq 0, \quad i, j = 1, \dots, n,$$

and in a neighborhood of $\zeta \in M$, the holomorphic functions of z

$$(4.2) \quad \mu_j(z) = \frac{\partial}{\partial \bar{\zeta}_j} \ln \frac{K_M(z, \bar{\zeta})}{K_M(\zeta, \bar{\zeta})}, \quad j = 1, \dots, n,$$

form a local system of coordinates, due to the condition (4.1). If we consider the mapping $f : M \rightarrow M$ given by a holomorphic transformation $(z, \zeta) \rightarrow (\tilde{z}, \tilde{\zeta})$, then we have

$$\mu_j(z) = \sum_{k=1}^n \tilde{\mu}_k(\tilde{z}) \frac{\partial \tilde{\zeta}_k}{\partial \zeta_j}, \quad j = 1, \dots, n,$$

i.e. any biholomorphic transformation of M can be described as a linear transformation of the coordinates μ_j , called *covariant representative coordinates in M* , see e.g. [65]. The following linear combinations of the coordinates μ_j (4.2) are considered

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = h^{-1} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

The Bergman representative coordinates relative to a point z_0 of a homogenous Kähler manifold M are defined by the formulae

$$(4.3) \quad w_i(z) = \sum_{j=1}^n h^{\bar{j}i}(z_0) \frac{\partial}{\partial \bar{\zeta}_j} \ln \frac{K_M(z, \bar{\zeta})}{K_M(\zeta, \bar{\zeta})} \Big|_{\zeta=z_0}, \quad i = 1, \dots, n,$$

where $h^{\bar{j}i}(z_0)$ is the inverse of the matrix $h_{i\bar{j}}(z_0)$ calculated with (2.20) from the kernel function K_M (2.16). This definition differs from the standard definition of the Bergman representative coordinates, where, instead of the kernel function K_M (2.16), it is used the Bergman kernel of the complex manifold $\mathcal{B}_n(z, \bar{w})$ defined in (5.8) and instead of the matrix h of the balanced metric (2.20) it is used the Bergman tensor (5.10).

The image of a bounded domain $\mathcal{D} \subset \mathbb{C}^n$ by the mapping $RC : (z_1, \dots, z_n) \rightarrow (w_1, \dots, w_n)$ was called by Bergman *representative domain* of \mathcal{D} , but we shall use the same denomination also for homogeneous Kähler manifolds. The mapping RC is generally holomorphic and one-to-one only locally. The name *representative coordinates* was firstly used by Fuks [36].

We also denote by $J(z_1, \dots, z_n)$ the matrix which determines the change of coordinates $RC : (z_1, \dots, z_n) \rightarrow (w_1, \dots, w_n)$ from local coordinates on M to the Bergman representative coordinates (4.3)

$$(4.4) \quad J(z_1, \dots, z_n) := \frac{\partial(w_1, \dots, w_n)}{\partial(z_1, \dots, z_n)}.$$

We have already underlined that

Remark 3.

$$(4.5) \quad J(z_1, \dots, z_n) \Big|_{z=z_0} = 1_n,$$

and in a neighborhood of z_0 the new coordinates (w_1, \dots, w_n) should yield a basis of local vector fields, i.e.

$$(4.6) \quad \det(J(z_1, \dots, z_n)) \neq 0.$$

In his paper [52], Lu Qi-Keng presented many examples of bounded domains $\mathcal{D} \subset \mathbb{C}^n$ in which $\mathcal{K}_{\mathcal{D}}(z, \bar{w}) \neq 0, \forall z, w \in \mathcal{D}$, see the standard definition in §5.1 in the Appendix. The domains which have a zero free Bergman kernel are called *domains satisfying the Lu Qi-Keng conjecture* or *Lu Qi-Keng domains*. It was conjectured in [52] that *any simply connected domain in \mathbb{C}^n is a Lu Qi-Keng domain*. Swarczyński provided an example of an unbounded Reinhardt domain $\mathcal{D} \subset \mathbb{C}^2$ for which $\mathcal{K}_{\mathcal{D}}(z, \bar{w})$ has a zero [64]. Later, a bounded, strongly pseudo-convex counterexample to Lu-Qi Keng conjecture was given by Boas [25].

It is known that [66]:

Remark 4. *The homogenous bounded domains are Lu Qi-Keng domains and consequently the associated Bergman representative coordinates are globally defined.*

In this paper we shall use the name *Lu Qi-Keng manifold* to denote a manifold which has properties similar to the Lu Qi-Keng domain, i.e. a homogenous Kähler manifold for which

$$(4.7) \quad K_M(z, \bar{w}) := (e_z, e_{\bar{w}}) \neq 0, \quad \forall z, w \in M.$$

Another name for a manifold M whose points $z \in M$ verify the condition $K_M(z) \neq 0$ would be *normal manifold*, a denomination used by Lichnerowicz for manifold for which $\mathcal{K}_M(z, \bar{z}) \neq 0$ or *Kobayashi manifold*, see §5.2.2 in Appendix.

We recall that in [3], in the context of Perelomov's CS, we have called the set $\Sigma_z := \{w \in M | K_M(z, \bar{w}) = 0\}$ the *polar divisor of $z \in M$* . We have shown that for *compact homogeneous manifolds for which the exponential map from the Lie algebra to the Lie group equals the geodesic exponential, and in particular for symmetric spaces, the set Σ_z has the geometric significance of the cut locus \mathbf{CL}_z* (see p. 100 in [45] for the definition of cut locus) attached to the point $z \in M$ [3]. Moreover, it was proved in [4] in the context of Rawnsley-Cahen-Gutt approach to Berezin's CS that if $\Sigma_z = \mathbf{CL}_z$, then indeed, Σ_z is a *polar divisor* in the meaning of algebraic geometry [40]. The case of conjugate locus in \mathbb{CP}^∞ as polar divisor in [3] is considered in [54], but it should be taken into consideration that in such a case the cut locus is identical with first conjugate point [31].

4.2. The simplest example: the Siegel disk. In order to have a feeling of what the Bergman representative coordinates are, we take the simplest example of the Siegel disk \mathcal{D}_1 with the Bergman kernel function (3.3).

Remark 5. *The Bergman kernel function (3.3) is positive definite and the Siegel disk \mathcal{D}_1 is a Lu Qi-Keng domain. The Bergman representative coordinates on the Siegel disk are global and the RC-transformation is a Kähler homogenous diffeomorphism.*

Proof. Indeed, the metric matrix $h(w) = \{h_{w\bar{w}}\}$ on \mathcal{D}_1 calculated in (3.5) is

$$(4.8) \quad h(w) = \frac{\partial^2}{\partial w \partial \bar{w}} \ln K_k(w, \bar{w}) = \frac{2k}{P^2}, \quad P = 1 - w\bar{w}.$$

We introduce in the formula (4.3) the data (3.3) and (4.8) and we get for the Bergman representative coordinate on the Siegel disk \mathcal{D}_1 the expression

$$(4.9) \quad w_1(w) = P_0 \frac{w - w_0}{P_1}, \quad P_0 = 1 - w_0\bar{w}_0, \quad P_1 = 1 - \bar{w}_0 w.$$

The Bergman representative coordinate for the Siegel disk has the right properties, i.e. $w_1(w_0) = 0$, and (4.5) and (4.6) are also verified because

$$(4.10) \quad \frac{\partial w_1}{\partial w} \Big|_{w_1 \in \mathcal{D}_1} = \frac{P_0}{P_1^2} \neq 0.$$

Note that the inverse of (4.9) is

$$(4.11) \quad w = \frac{w'_1 + w_0}{1 + \bar{w}_0 w'_1}, \quad w'_1 = \frac{w_1}{P_0},$$

and

$$(4.12) \quad K_k(w'_1, \bar{w}'_1) = \left(\frac{1 - |w'_1|^2}{|1 + \bar{w}_0 w'_1|^2} \right)^{-2k}.$$

Also we obtain the expression

$$(4.13) \quad -i\omega_k(w'_1) = 2k \frac{d w'_1 \wedge d \bar{w}'_1}{(1 - w'_1 \bar{w}'_1)^2}.$$

□

4.3. The Siegel-Jacobi disk. We consider the Siegel-Jacobi disk endowed with the kernel function (3.7). We proof that:

Proposition 3. *The kernel function of the Siegel-Jacobi disk (3.7) is positive definite, \mathcal{D}_1^J is a normal Kähler homogeneous Lu Qi-Keng manifold. The Bergman representative coordinates (4.3) are globally defined on \mathcal{D}_1^J .*

Proof. Using (3.40), we particularize (4.3) to \mathcal{D}_1^J :

$$\begin{aligned} w_1(\varsigma) &= \left(h^{\bar{1}1}(\varsigma) \frac{\partial}{\partial \bar{z}'} + h^{\bar{2}1}(\varsigma) \frac{\partial}{\partial \bar{w}'} \right) X, \\ w_2(\varsigma) &= \left(h^{\bar{1}2}(\varsigma) \frac{\partial}{\partial \bar{z}'} + h^{\bar{2}2}(\varsigma) \frac{\partial}{\partial \bar{w}'} \right) X, \end{aligned}$$

where, with the notation of (3.7) and (3.8), we have

$$(4.14) \quad X := X(\varsigma, \varsigma') = \ln \frac{K_{k\mu}(\varsigma, \varsigma')}{K_{k\mu}(\varsigma')}. \quad \square$$

We find explicitly the representative Bergman coordinates (4.3) $(w_1, w_2) = RC(z, w)$ on the Siegel-Jacobi disk \mathcal{D}_1^J

$$(4.15) \quad \begin{aligned} w_1(\varsigma) &= -\bar{\eta}_0 w_2 + P_0(\eta'_1 - \eta_0), \\ w_2(\varsigma) &= \frac{P_0}{P_1}(w - w_0) + \lambda [P_0(\eta'_1 - \eta_0)]^2, \end{aligned}$$

where

$$(4.16) \quad \eta_0 := \eta(z_0, w_0) = \frac{z_0 + \bar{z}_0 w_0}{P_0}, \quad \eta'_1 = \frac{z + \bar{z}_0 w}{P_1}, \quad \lambda = \frac{\mu}{4k}.$$

We have

$$(w_1(\varsigma_0), w_2(\varsigma_0)) = (0, 0).$$

Note that the Bergman representative coordinates (w_1, w_2) (4.15), (4.16) verify the condition (4.4), i.e.

$$(4.17) \quad J(\varsigma)|_{\varsigma=\varsigma_0} = 1_2.$$

We have also

$$(4.18) \quad \det J(\varsigma)|_{\varsigma \in \mathcal{D}_1^J} = \left(\frac{P_0}{P_1} \right)^3 \neq 0.$$

Now we reverse equation (4.15) and we express $\varsigma = (z, w) \in \mathbb{C} \times \mathcal{D}_1$ of a point in \mathcal{D}_1^J as function of the Bergman representative coordinates (w_1, w_2) and we get:

$$(4.19a) \quad w = \frac{y + w_0}{Q},$$

$$(4.19b) \quad z = \frac{z_0 - \bar{z}_0 y + P_0 x}{Q},$$

where

$$(4.20) \quad Q = 1 + \bar{w}_0 y, \quad y = w'_2 - \lambda P_0 x^2, \quad x = w'_1 + \bar{\eta}_0 w'_2; \quad w'_i = \frac{w_i}{P_0}, \quad i = 1, 2.$$

If we put in (4.19) $(w_1, w_2) = (0, 0)$, then $(w, z) = (w_0, z_0)$. \square

Now we express the Kähler two-form (3.23) on the Siegel disk \mathcal{D}_1^J in the Bergman representative coordinates (w_1, w_2) (4.15). The reverse to the transformation $(x, y) \rightarrow (z, w)$ given by equations (4.19) is

$$(4.21) \quad x = \frac{z - \eta_0 + \bar{\eta}_0 w}{P_1}, \quad y = \frac{w - w_0}{P_1}.$$

We shall prove that

Proposition 4. *The RC-transformation (4.21) $(w, z) \rightarrow (x, y)$ is a biholomorphic mapping of the Siegel-Jacobi disk to itself. The Kähler two-form $\omega_{k\mu}(z, w)$ of the Siegel-Jacobi disk expressed in the variables (x, y) from (4.19) has a value similar to (3.23):*

$$(4.22) \quad -i\omega_{k\mu}(x, y) = 2k \frac{dy \wedge d\bar{y}}{(1 - |y|^2)^2} + \mu \frac{\mathcal{B} \wedge \bar{\mathcal{B}}}{1 - |y|^2},$$

where the one-form \mathcal{B} is defined in (4.27d), (4.27e). The action of an element $(l, u) \in \text{SU}(1, 1) \times \mathbb{C}$ on $(x, y) \in \mathcal{D}_1^J$, $(l, u) \circ (x, y) = (x', y')$, where

$$(4.23) \quad l = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix}, \quad |r|^2 - |s|^2 = 1,$$

is similar to the action (3.6), (3.9), i.e.

$$(4.24) \quad x' = \frac{x + u - \bar{u}y}{\bar{s}y + \bar{r}}, \quad y' = \frac{ry + s}{\bar{s}y + \bar{r}},$$

where the parameters of the matrix $l \in \text{SU}(1, 1)$ and $u \in \mathbb{C}$ are expressed as function of the matrix $g \in \text{SU}(1, 1)$ (3.6) and $\alpha \in \mathbb{C}$ describing the action $(g, \alpha) \circ (w, z) = (w_1, z_1)$ in (3.6), (3.9):

$$(4.25a) \quad r = \frac{a - \bar{b}w_0 + \bar{w}_0(b - w_0\bar{a})}{P_0},$$

$$(4.25b) \quad s = \frac{b + w_0(a - \bar{a} - w_0\bar{b})}{P_0},$$

$$(4.25c) \quad u = \frac{z_0 + \alpha - w_0\bar{\alpha} - \eta_0(\bar{a} + w_0\bar{b}) + \bar{\eta}_0(b + w_0a)}{P_0}.$$

The Kähler two-form (4.22) $\omega_{k\mu}(x, y)$ is Kähler homogenous, i.e. it is invariant to the action (4.24). The RC-transformation (4.21) for the Siegel-Jacobi disk is a homogenous Kähler diffeomorphism and the representative manifold of the Siegel-Jacobi disk is the Siegel-Jacobi disk itself.

We express the resolution of unit on the Siegel-Jacobi disk (3.10) in the variables (x, y) related to the variables (z, w) by (4.19). The measure (3.12), G_1^J -invariant to the

action (4.24), has the expression:

$$(4.26) \quad d\nu(x, y) = \mu \frac{d\Re y d\Im y}{(1 - y\bar{y})^3} d\Re x d\Im x.$$

We express the reproducing kernel (3.8) in the variables (x, y) given by (4.20). For P we have the expression (4.27a), while for $F(x, y)$ we have the expression:

$$\begin{aligned} 2(1 - |y|^2)|Q|^2 F(x, y) &= |y|^4(w_0\bar{\eta}_0\bar{z}_0 + cc) + |y|^2(yx_1 + cc) + P_0(\bar{w}_0\bar{x}^2y^2 + cc) \\ &\quad + y[-x_1 + P_0\bar{x}(2\bar{w}_0x + \bar{x}(1 + |w_0|^2)) + cc \\ &\quad + |y|^2[|z_0|^2 - \eta_0(\bar{z}_0 + P_0\bar{x}) - cc] + \eta_0 + x(1 + w_0) + cc, \\ \text{where } x_1 &= \bar{z}_0^2 - 2\bar{w}_0\eta_0P_0\bar{x}. \end{aligned}$$

Proof. In order to do this calculation, we use the values of (z, w) from (4.19) and we have

$$(4.27a) \quad P = 1 - w\bar{w} = P_0 \frac{1 - |y|^2}{|Q|^2},$$

$$(4.27b) \quad dw = P_0 \frac{dy}{Q^2},$$

$$(4.27c) \quad dz = P_0 \frac{-(\bar{\eta}_0 + \bar{w}_0x)dy + Qdx}{Q^2},$$

$$(4.27d) \quad \eta(z, w) = \eta_0 + \frac{B + w_0\bar{B}}{1 - |y|^2}, \quad B := x + \bar{x}y,$$

$$(4.27e) \quad \mathcal{A} = P_0 \frac{\mathcal{B}}{Q}, \quad \mathcal{B} = dx + \bar{\eta}(x, y)dy, \quad \eta(x, y) = \frac{B}{1 - |y|^2}.$$

From (4.27a), we find that $w \in \mathcal{D}_1 \iff y \in \mathcal{D}_1$, i.e.

$$1 - |w|^2 > 0 \iff 1 - |y|^2 > 0.$$

Note that the change of coordinates $(z, w) \rightarrow (w_1, w_2) \rightarrow (y, x)$ is well defined on \mathcal{D}_1^J because

$$\frac{\partial(y, x)}{\partial(z, w)} = \frac{\partial(y, x)}{\partial(w_1, w_2)} \frac{\partial(w_1, w_2)}{\partial(z, w)},$$

and

$$\det \frac{\partial(y, x)}{\partial(w_1, w_2)} = \frac{1}{P_0},$$

so, with (4.18), we have

$$\det \frac{\partial(y, x)}{\partial(z, w)} \Big|_{\mathcal{D}_1^J} = \frac{P_0^2}{P_1^3} \neq 0.$$

Taking into account (4.27a), (4.27b) and (4.27c), we get for the measure (3.12) the expression (4.26). \square

5. APPENDIX

5.1. Bergman pseudometric and Bergman metric. Let M be a complex n -dimensional manifold. We consider three Hilbert spaces, denoted $\mathfrak{F}_\mathfrak{H}$, $\mathcal{F}_2(M)$ and $\mathcal{F}_n(M)$, and, under some conditions, we shall establish correspondences between them.

Let $M = G/H$ be a homogenous Kähler manifold. We consider the Hilbert space $\mathfrak{F}_\mathfrak{H}$ with scalar product (2.15), a particular realization of weighted Hilbert space \mathfrak{H}_f (2.4) corresponding to a constant ϵ -function (2.19), endowed with the base of functions $\{\varphi_i\}_{i=0,1,\dots}$, verifying (2.17). To the Kähler potential $f = \ln(K_M)$ we associate the balanced metric (2.20).

For a n -dimensional complex manifold M , let us denote by $\mathcal{F}_2(M)$ the Hilbert space of square integrable functions with respect to the scalar product

$$(5.1) \quad (f, g)_{\mathcal{F}_2(M)} = i^{n^2} \int_M \bar{f}(z) g(z) \, dz \wedge d\bar{z} = \int_M \bar{f}(z) g(z) \, dV,$$

where we have introduced the abbreviated notation $dz = dz_1 \wedge \dots \wedge dz_n$ and dV has the expression given in (2.13). This space was considered by Stefan Bergman for bounded domains $\mathcal{D} \subset \mathbb{C}^n$ [20, 21, 22].

Let $\{\Phi_i\}_{i=0,1,\dots}$ be a orthonormal basis of $\mathcal{F}_2(M)$. Then to the *Bergman kernel function*

$$(5.2) \quad \mathcal{B}_2(z, \bar{w}) = \sum_{i=0}^{\infty} \Phi_i(z) \bar{\Phi}_i(w),$$

it is associated the *Bergman metric*

$$(5.3) \quad ds_{\mathcal{B}_2}^2(z) = \sum_{i,j=1}^n \frac{\partial^2 \ln \mathcal{B}_2(z, \bar{z})}{\partial z_i \partial \bar{z}_j} \, dz_i \otimes d\bar{z}_j.$$

Let now denote by $\mathcal{F}_n(M)$ the set of square integrable holomorphic n -forms on the complex n -dimensional manifold M , i.e.

$$(5.4) \quad i^{3n^2} \int_M \bar{f} \wedge f < \infty.$$

This space was considered by Weil [67], Kobayashi [43, 46], Lichnerowicz [49], while for homogeneous manifolds, see Koszul [48] and Piatetski-Shapiro [58, 59]. *The vector space $\mathcal{F}_n(M)$ is a separable complex Hilbert space with countable base* (cf Corollaries at p. 60 in [67]) and if we write $f = f^* dz$, $g = g^* dz$, then the Hilbert space has the inner product given by

$$(5.5) \quad (f, g)_{\mathcal{F}_n(M)} = i^{3n^2} \int_M \bar{f} \wedge g = \int_M \bar{f}^* g^* \, dV,$$

which is invariant to the action of holomorphic transformations of M (cf the Theorem at p. 353 in [49]).

Let $f_0(z), f_1(z), \dots$ be a complete orthonormal basis of the space $\mathcal{F}_n(M)$ of square integrable holomorphic n -forms on M . It can be shown (cf Theorem 1 at p. 357 in [49]) that the series

$$\sum_{k=0}^{\infty} f_k(z) \wedge \bar{f}_k(w)$$

is absolutely convergent and defines a form $\mathcal{K}_M(z, \bar{w})$ holomorphic in z and \bar{w} ,

$$(5.6) \quad \mathcal{K}_M(z, \bar{w}) = \sum_{k=0}^{\infty} f_k(z) \wedge \bar{f}_k(w),$$

called the *Bergman kernel form of the manifold M* , which is independent of the base and is invariant under the group of holomorphic transformations of M [43, 49].

For $f_i(z) \in \mathcal{F}_n(M)$, let us denote by $f_i^*(z)$ the holomorphic function defined locally such that

$$(5.7) \quad f_i(z) = f_i^*(z) \, d z.$$

Then we have the relation

$$(5.8) \quad \mathcal{K}_M(z, \bar{w}) = \mathcal{B}_n(z, \bar{w}) \, d z \wedge d \bar{w},$$

where the *Bergman kernel* $\mathcal{B}_n(z, \bar{w})$ of the complex manifold M (as it is called by Lichnerowicz in [49]) has the series expansion (5.9)

$$(5.9) \quad \mathcal{B}_n(z, \bar{w}) = \sum_{i=0}^{\infty} f_i^*(z) \bar{f}_i^*(w).$$

A complex analytic manifold M for which $\mathcal{K}_M(z, \bar{z})$ is different of zero in every point of $z \in M$ is called by Lichnerowicz *normal manifold* (cf p. 367 in [49]). The covariant symmetric tensor t of type $(1, 1)$, called the *Bergman tensor of the complex normal manifold M* , has locally the components

$$(5.10) \quad t_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\beta}} \ln(\mathcal{B}_n(z, \bar{z})), \quad \alpha, \beta = 1, \dots, n.$$

The *Bergman tensor of a complex normal manifold M* is invariant under all holomorphic transformations of the manifold M (cf the Theorem at p. 369 in [49]).

Assume that we are in the points $z \in M' \subset M$ where $\mathcal{B}_n(z, \bar{z}) > 0$ and let us define the Bergman pseudometric (in the sense of [42])

$$(5.11) \quad d s_{\mathcal{B}_n}^2(z) = \sum_{i,j=1}^n t_{i\bar{j}} \, d z_i \otimes d \bar{z}_j.$$

Following §5 and §8 in [49], for every $z \in M$, let us introduce the subspace of $\mathcal{F}_n(M)$

$$\mathcal{F}'_n(z) = \{\gamma \in \mathcal{F}_n(M) | \gamma(z) = 0\}.$$

Let $\alpha_0 \in \mathcal{F}_n(M)$ such that

$$(5.12) \quad \alpha_0(z) \neq 0, \quad (\alpha_0, \alpha_0)_{\mathcal{F}_n(M)} = 1, \quad (\alpha_0, \gamma)_{\mathcal{F}_n(M)} = 0, \quad \forall \gamma \in \mathcal{F}'_n(z).$$

An *adopted orthonormal basis* of $\mathcal{F}_n(M)$ is a basis $\alpha_0, \alpha_1, \dots$ such that α_0 verifies (5.12) and $\alpha_i(z) = 0$ for $i > 0$. Then

$$\mathcal{K}_n(z, \bar{z}) = \alpha_0(z) \wedge \bar{\alpha}_0(z), \quad \mathcal{B}_n(z, \bar{z}) = |\alpha_0(z)|^2 > 0 \quad \text{if } \mathcal{F}'_n(z) \neq \mathcal{F}_n(M),$$

and we get

$$d s_{\mathcal{B}_n}^2(z) = \frac{1}{\mathcal{B}_n(z, \bar{z})} \sum_{k=1}^{\infty} |d \alpha_k(z)|^2 > 0,$$

i.e. $ds_{\mathcal{B}_n}^2(z)$ is positive definite in M' .

In fact, we have the assertions: *The quadratic form (5.11) is positive semi-definite in the points of $M' \subset M$ and invariant under holomorphic transformations in M and Kählerian. For Kobayashi manifolds the Bergman metric (5.11) is positive definite. Bounded domains in \mathbb{C}^n have positive definite Bergman metric, see Theorem 3.1 in [43], Ch III in [67] and §3.2 in [39].*

Now we establish correspondences of the basis of the Hilbert spaces denoted $\mathfrak{F}_{\mathfrak{H}}$, $\mathcal{F}_2(M)$, and $\mathcal{F}_n(M)$, comparing respectively the formulae (2.15), (5.1) and (5.5), in the situation that this is possible.

Proposition 5. *Let M be a n -dimensional complex manifold M . We have the identity*

$$(5.13) \quad \mathcal{B}_n(z, \bar{w}) = \mathcal{B}(z, \bar{w}), \text{ where } \mathcal{B}(z, \bar{w}) := \mathcal{B}_2(z, \bar{w}).$$

The Bergman metrics (5.3) and (5.11) coincides

$$(5.14) \quad ds_{\mathcal{B}_2}^2 = ds_{\mathcal{B}_n}^2 = ds_{\mathcal{B}}^2, \text{ where } ds_{\mathcal{B}}^2 = \sum_{i,j=1}^n \frac{\partial^2 \mathcal{B}(z, \bar{z})}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j,$$

but they are different from the metric $ds_M^2(z)$ given by (2.20).

Let now $M = G/H$ be a homogeneous Kähler manifold. We have also the correspondence of the base functions (5.7) $f_i \leftrightarrow f_i^$ defining $\mathcal{F}_n(M)$, Φ_i defining $\mathcal{F}_2(M)$, and φ_i defining $\mathfrak{F}_{\mathfrak{H}}$*

$$(5.15) \quad f_i^* \leftrightarrow \Phi_i \leftrightarrow \sqrt{\Upsilon} \varphi_i, \quad i = 0, 1, \dots, \quad \Upsilon = \frac{\mathcal{G}(z)}{K_M(z)}.$$

On homogeneous manifolds, the Bergman $2n$ -form (5.8) can be expressed in function of the normalized Bergman kernel (2.23) by the formula

$$(5.16) \quad \mathcal{B}_n(z, \bar{w}) = \sqrt{\Upsilon(z)\Upsilon(w)} K_M(z, \bar{w}) = \sqrt{\mathcal{G}(z)\mathcal{G}(w)} \kappa_M(z, \bar{w}).$$

In particular, we have the relations:

$$(5.17) \quad \mathcal{B}_n(z, \bar{z}) \equiv \mathcal{G}(z).$$

If ω_M^1 is the Kähler two-form associated to the Kähler potential $\mathcal{B}(z, \bar{z})$, then

$$(5.18) \quad \omega_M^1 = i\partial\bar{\partial} \ln(\mathcal{G}) = -\rho_M,$$

where ρ_M denotes the Ricci form (3.19) and the homogenous manifold M is Einstein with respect to the Bergman metric $ds_{\mathcal{B}}^2$.

5.2. Embeddings. Let M (N) be a complex manifold of dimension m (respectively, n). The continuous mapping $f : M \rightarrow N$ is called *holomorphic* if the coordinates of the image point are expressed as holomorphic functions of those of the original point. f is an *immersion* if $m \leq n$ and if the functional matrix is of rank m in all points of M . An immersion is an *embedding* if $f(x) = f(y)$, for $x, y \in M$ implies $x = y$, see e.g. p. 60 in [30].

It can be proved (see Theorem (E) at p. 60 in [30]): *Let N be a Kählerian manifold and let $f : M \rightarrow N$ be a holomorphic immersion. Then M has a Kählerian structure.*

We are concerned with manifolds M which admits an embedding in some projective Hilbert space attached to a holomorphic line bundle $\mathcal{L} \rightarrow M$

$$(5.19) \quad \iota : M \hookrightarrow \mathbf{P}(\mathcal{L}).$$

5.2.1. *The compact case.* Firstly, we recall the case of compact manifolds M .

1). A holomorphic line bundle \mathcal{L} on a compact complex manifold M is said *very ample* [63] if:

C₁) the set of divisors is without base points, i.e. there exists a finite set of global sections $s_1, \dots, s_N \in \Gamma(M, \mathcal{L})$ such that for each $m \in M$ at least one $s_j(m)$ is not zero;

C₂) the holomorphic map $\iota_{\mathcal{L}} : M \hookrightarrow \mathbb{CP}^{N-1}$ given by

$$(5.20) \quad \iota_{\mathcal{L}} = [s_1(m) : \dots : s_N(m)]$$

is a holomorphic embedding.

So, $\iota_{\mathcal{L}} : M \hookrightarrow \mathbb{CP}^{N-1}$ is an embedding (cf [40]) if the following conditions are fulfilled:

\mathring{A}_1) the set of divisors is without base points;

\mathring{A}_2) the differential of the map ι is nowhere degenerate;

\mathring{A}_3) the map ι is one-one, i.e. for any $m, m' \in M$ there exists $s \in \Gamma(M, \mathcal{L}) = H^0(M, \mathcal{O}(\mathcal{L}))$ such that $s(m) = 0$ and $s(m') \neq 0$, cf Proposition 4 § 22 p. 215 in Ref. [62].

5.2.2. *Kobayashi embedding.* Now we discuss the construction of the embedding (5.19) for noncompact complex manifolds M . Then the projective Hilbert space in (5.19) is infinite dimensional [43].

Following Kobayashi [43], we recall the conditions in which the Bergman pseudometric (5.11) is a metric on complex manifolds M .

The analogous of conditions \mathring{A}_1)- \mathring{A}_3) used by Kobayashi in the noncompact case are:

A₁) for any $z \in M$, there exists a square integrable n -form α such that $\alpha(z) \neq 0$, i.e. the Kernel form $\mathcal{K}_M(z, \bar{z})$ is different from zero in any point of M ;

A₂) for every holomorphic vector Z at z there exists a square integrable n -form f such that $f(z) = 0$ and $Z(f^*) \neq 0$, where $f = f^* dz_1 \wedge \dots \wedge dz_n$;

A₃) if z and z' are two distinct points of M , then there is a n -form f such that $f(z) = 0$ and $f(z') \neq 0$. The bounded domains in \mathbb{C}^n have also more specific properties, as C -hyperbolicity and hyperbolicity in the sense of Kobayashi [39].

Piatetski-Shapiro calls a manifold with properties A₁)-A₂) a *Kobayashi manifold* [58]. The main property of the Kobayashi manifolds similar with that of bounded domains in \mathbb{C}^n is the existence of a *positive definite* Bergman metric, which we present below.

Let α be the holomorphic n -form defined at A₁). Then to any $f \in \mathcal{F}_n(M)$ we put into correspondence an analytic function $h(z)$ defined by the relation $f = h\alpha$. The set of such functions generates a Hilbert space of functions \mathcal{F} with scalar product

$$(5.21) \quad (h, h') = i^{3n^2} \int_M \bar{h} h' \bar{\alpha} \wedge \alpha,$$

isomorphic with $\mathcal{F}_n(M)$. We see below that under some conditions this space \mathcal{F} can be identified with the Hilbert space $\mathfrak{H}_{\mathfrak{H}}$ with the scalar product (2.15), see Remark 6.

Then $\mathcal{L} = \mathcal{F}^*$. Let $z = (z_1, \dots, z_n)$ be a local coordinate system. Let ι' be the mapping which sends z into an element $\iota'(z)$ of \mathcal{L} defined by the pairing $\langle \iota'(z), f \rangle =$

$f^*(z)$, where $f(z_1, \dots, z_n) = f^* d z_1 \wedge \dots \wedge d z_n \wedge d \bar{z}_1 \wedge \dots \wedge d \bar{z}_n$. Then $\iota'(z) \neq 0$ if a condition analogous to condition \tilde{A}_1 in the noncompact case is satisfied. Then $\iota = \xi \circ \iota'$ is independent of local coordinates and is continuous and complex analytic.

Note that if M satisfies A_3 , then it automatically satisfies A_1).

Kobayashi has shown (see theorems 3.1, 8.1, 8.2 in [43]) that

Proposition 6. *Let M be a complex manifold. Condition A_1 implies that the quadratic form (5.11) is positive semi-definite, invariant under the holomorphic transformations of M , $ds_{\mathcal{B}_n}^2$ is induced from the canonical Fubini-Study Kähler metric (2.27) and we have a relation similar to (2.30):*

$$(5.22) \quad ds_{\mathcal{B}_n}^2 = \iota^*(ds_{FS}^2).$$

The differential $d\iota$ is not singular at any point of M if and only if M satisfies A_2). If M satisfies A_1 and A_2 , then ι is an isometric immersion of M into \mathbb{CP}^∞ . If M is a complex manifold satisfying A_2 and A_3 , then ι is an isometrical imbedding of M into \mathbb{CP}^∞ and we have (5.22).

Remark 6. *Let us suppose that the homogeneous Kähler manifold M is such that we have the space $\mathfrak{F}_5 \neq 0$ with positive definite kernel function (2.16) and M is a Lu Qi-Keng manifold. The Bergman kernel form (5.6) and the Bergman kernel function are related by (5.16).*

In conclusion, in this paper we have illustrated elements of Berezin's quantization on the partially bounded manifold \mathcal{D}_1^J . Propositions 1 and 2 have been already published [15], but in the present paper we added new assertions and statements. We have considered representative Bergman coordinates on Lu Qi-Keng and normal manifolds. The Propositions 3, 4, 5 and the Remark 6 contain the main results of this paper.

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